Brace for anomalies in soap bubbles

When you dip a wire frame into soapy water, it forms a soap film spanning the wire frame. By physical laws, the soap film is a liquid surface that minimizes the area — among all surfaces spanning the same boundary frame — and thus it is called a minimal surface. The problem of looking for a surface with least area (or sometimes least surface tension or energy) also manifests in other physical problems, such as determining the shape of a bridge constructed of the least amount of materials; the structure of liquid crystals used in LCDs; and the event horizon near a black hole.

Most minimal surfaces we encounter in everyday life appear to be smooth. But what if we run this experiment using a wire frame with a quirky shape, such as a wire that winds around more than once, or one that crosses itself? Do we still see a nice smooth soap film? This question — and its generalization to higher dimensions — has intrigued mathematicians since the eighteenth century. The first Fields Medal, the most coveted prize/highest honor in mathematics, was awarded to Jesse Douglas in 1936, for his contribution to this field of research.

Figure 1 Soap films with singularities at corners and edges

The reality that we live in a small 3-dimensional world is why we usually see smooth minimal surfaces without quirks. But to creatures living in a 4-dimensional world or higher, their soap bubbles exhibit a much richer variety of shapes. This is because in higher dimensions, the soap film has a lot more room to twist and turn and form interesting shapes — and singularities.

Since a singularity is often inevitable, mathematicians set out to study how big the set of such problematic points can be: Can they appear everywhere on the surface, or are they quite limited? And is it possible to tweak the wire frame to avoid singularities?

To understand these questions, one needs to use a magnifying glass to zoom in at places of singularity, and see what the soap film looks like nearby. It turns out that when one looks closely enough, all singular points look like cones. Therefore, the question becomes: What possible cones could arise that minimize the area? And, knowing which cone it is, can we recover the shape of the minimal surface that looks like this cone under an infinitely-powerful magnifying glass?

This is the roadmap for understanding all possible singularities of minimal surfaces. Similarly, when we look at a curvy road on a map, it is natural to first tilt our heads to the direction of the road (which is a straight line that plays the same role as the cone), and then look at how far the curvy road deviates from this straight line. There may even be singularities along the road, where it takes a sharp turn — in which case the direction of the road is no longer a straight line, but a cone (with the turning angle). The roadmap to analyzing singularities of minimal surfaces is an analogous process; but complexity arises since we are looking at a surface in higher dimensions, instead of on a planar map, and thus there are more directions and ways that the minimal surface can turn.

Nonetheless, just as a curvy road can be written as the graph of a function over the horizon, a minimal surface can be written as the graph of a function over its model cone. The condition that it minimizes the area, is then reduced to the property that this function satisfies certain differential equations. This way, the geometric problem of looking for singularities is reduced to a problem of solving differential equations.

Following this roadmap, and by accumulating the work of several generations of mathematicians, we now have a fairly good understanding about the singularities of minimal surfaces. In particular, although points of singularity are usually unavoidable, they only form a very small set whose total area is zero. If an alien living in a 4-dimensional world blows a soap bubble, the bubble has non-smooth points, but they can be counted out and they are never too close to each other.